Well-posedness of Non-autonomous Linear Evolution Equations in Uniformly Convex Spaces

Jochen Schmid and Marcel Griesemer
Fachbereich Mathematik, Universität Stuttgart, D-70569 Stuttgart, Germany
firstname.lastname@mathematik.uni-stuttgart.de

Abstract

This paper addresses the problem of well-posedness of non-autonomous linear evolution equations $\dot{x} = A(t)x$ in uniformly convex Banach spaces. We assume that $A(t) : D \subset X \to X$ for each t is the generator of a quasi-contractive, strongly continuous group, where the domain D and the growth exponent are independent of t. Well-posedness holds provided that $t \mapsto A(t)y$ is Lipschitz for all $y \in D$. Hölder continuity of degree $\alpha < 1$ is not sufficient and the assumption of uniform convexity cannot be dropped.

1 Introduction

In the literature the existence of the propagator (evolution system) for the non-autonomous Schrödinger equation is often discussed within the more general context of abstract non-autonomous linear evolution equations

$$\dot{x} = A(t)x, \qquad x(s) = y \tag{1}$$

in some Banach space X, where $A(t) : D(A(t)) \subset X \to X$ for each $t \in [0, T]$ is the generator of a strongly continuous semigroup, $0 \leq s \leq t$ and $y \in D(A(s))$. On the level of proofs this approach involves serious technical difficulties that are associated with the lack of structure of general Banach spaces and the non-reversibility of the dynamics given by a semigroup. The prize for the solution of these problems is paid in terms of regularity assumptions on $t \mapsto A(t)$ [7, 8, 2, 16, 10, 4].

In the present paper, which is motivated by the Schrödinger equation, the evolution problem (1) is discussed in a more restrictive setting, which does not have the drawbacks mentioned above. In this setting X is a uniformly convex Banach space and $A(t)$ for each $t \in [0, T]$ is the generator of a strongly continuous *group* rather than a semigroup. We assume, moreover, that this group is quasi-contractive with a growth exponent that is independent of t and that the domain $D = D(A(t))$ is independent of t as well. Our main result, in the simplest form, establishes the existence of a unique evolution system $U(t, s)$ provided

$$t \mapsto A(t)y \tag{2}$$

is Lipschitz for all $y \in D$. It follows that $t \mapsto x(t) = U(t, s)y$ is the unique continuously differentiable solution of (1) and that it depends continuously on the initial data s and y (well-posedness). We give examples showing that Hölder continuity of the map (2) is not sufficient and that Lipschitz continuity is not sufficient anymore if the assumption

of uniform convexity is dropped. This means in particular that well-posedness of the non-autonomous Schrödinger equation, that is, Equation (1) with X a Hilbert space and $A(t)^* = -A(t)$, requires less regularity than well-posedness of (1) in the general Banach space setting.

The well-posedness of (1) in uniformly convex Banach spaces was previously studied by Kato [6, 7]. Our result described above could be derived, with some work, from Theorem 5.2 combined with the information from the Remark 5.3 in [7]. See Theorem 3.2 of [17] for a Hilbert space version of Kato's Remark 5.3 in [7]. Our main result, Theorem 2.1 below, does not follow from Kato's work but it reduces to a theorem of Kato if X is a Hilbert space and $A(t)^* = -A(t)$, see Theorem 3 of [6] [1]. To explore the necessity of our assumptions, we give new counterexamples to well-posedness that are of a very simple and transparent type. Last but not least, the present paper shows that the essence of Kato's work in the uniformly convex case can be summarized in a short and simple proof that requires nothing but basic functional analysis and a rudimentary knowledge of semigroup theory.

2 Results and Examples

Let X be a complex Banach space and let $A(t) : D \subset X \to X$ for $t \in [0, T]$ be a family of closed linear operators with a time-independent dense domain $D \subset X$. A two-parameter family of linear operators $U(t, s) \in \mathscr{L}(X)$, will be called an *evolution system for* $A(t)$ *on* D if the following conditions are satisfied:

(i) $U(t, s)D \subset D$ and the map $t \mapsto U(t, s)y$ on $[0, T]$ is a continuously differentiable solution of (1) for any $y \in D$ and $s \in [0, T]$.

(ii) $U(s, s) = 1$ and $U(t, r)U(r, s) = U(t, s)$ for all $s, r, t \in [0, T]$.

(iii) $(t, s) \mapsto U(t, s)$ strongly continuous on $[0, T] \times [0, T]$.

Any two-parameter family of linear operators $U(t, s) \in \mathscr{L}(X)$ satisfying (ii) and (iii) is called an *evolution system*.

Existence of an evolution system $U(t, s)$ with the properties analogous to (i)-(iii) on the triangle $0 \le s \le t \le T$ is equivalent to well-posedness in the classical sense of C^1-solutions [3], Proposition VI.9.3. Our assumptions on $A(t)$ in Theorem 2.1 will allow us to construct $U(t, s)$ on the entire square $[0, T] \times [0, T]$ and this is essential for our proof.

For the reader's convenience we recall that a Banach space X is called uniformly convex if, given $\varepsilon > 0$ there exists $\delta > 0$ such that any pair of normalized vectors $x, y \in X$ with $\|(x + y)/2\| > 1 - \delta$ satisfies $\|x - y\| < \varepsilon$. Every Hilbert space and every $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ is uniformly convex. Uniformly convex Banach spaces are reflexive (Milman) and uniform convexity implies that weak convergence $x_n \rightharpoonup x$ turns into strong convergence as soon as $\|x_n\| \to \|x\|$.

As a final preparation we recall from [10, 3] that the norm of every strongly continuous semigroup e^{At}, $t \ge 0$, satifies a bound of the form $\|e^{At}\| \le Me^{\omega t}$. It follows that $\sigma(A) \subset \{\operatorname{Re} z \le \omega\}$. If $M = 1$ then the semigroup is called quasi-contractive.

[1] This result of Kato seems to have been overlooked by many later authors.

Theorem 2.1. *Let X be a uniformly convex Banach space and let $A(t) : D \subset X \to X$ for each $t \in [0, T]$ be the generator of a strongly continuous group with*

$$\|e^{A(t)s}\| \leq e^{\omega|s|}, \qquad s \in \mathbb{R}, \tag{3}$$

where ω and the domain D are independent of t. Suppose that $t \mapsto A(t) \in \mathscr{L}(Y, X)$ is continuous and of bounded variation, where Y is the space D endowed with the graph norm of $A(0)$. Then there exists a unique evolution system $U(t, s)$ for $A(t)$ on D.

Remark 2.2. The regularity assumption on $t \mapsto A(t) \in \mathscr{L}(Y, X)$ is clearly satisfied if this map is Lipschitz, which, by the principle of uniform boundedness, is equivalent to $t \mapsto A(t)y$ being Lipschitz for all $y \in D$.

Remark 2.3. Theorem 2.1 is false if the assumption of uniform convexity is dropped (Example 1), and moreover, even if X is a Hilbert space and $A(t)$ is skew-selfadjoint, the Lipschitz continuity cannot be replaced by Hölder continuity of some degree $\alpha < 1$ (Example 2). This is in sharp contrast to the case of parabolic evolution equations, where Hölder continuity is sufficient [10, 12].

Proof. Let $\underline{A}(t) := A(t) - (\omega + 1)$ and define $\|y\|_t := \|\underline{A}(t)y\|$ for $y \in D$. This norm is equivalent to the graph norm of $\underline{A}(t)$ and hence $Y_t = (Y, \|\cdot\|_t)$ is a Banach space. Like X, the space Y_t is uniformly convex as can be easily verified using the definition of uniform convexity given above. From now on $Y = Y_0$, which amounts to a different but equivalent choice of norm, compared to the definition of Y in the theorem.

In the special case where X is a Hilbert space and $A(t)$ is skew-selfadjoint, it follows that $\|y\|_t^2 = \|A(t)y\|^2 + \|y\|^2$ and hence Y_t is a Hilbert space too.

For $s, t \in [0, T]$ with $s < t$ we set

$$V(s, t) := C \sup \sum_{i=1}^{m} \|A(t_i) - A(t_{i-1})\|_{Y,X} \tag{4}$$

where the supremum is taken with respect to all partitions $s = t_0 < t_1 \ldots < t_m = t$ of the interval $[s, t]$. Up to the constant $C > 0$, which will be chosen later, $V(s, t)$ is the variation of $\tau \mapsto A(\tau)$ over the interval $[s, t]$. Let $V(t, s) := V(s, t)$. Apart from the obvious inequality $V(t, s) \geq C\|A(t) - A(s)\|_{Y,X}$, the properties of V that will be used in the following are, first, that

$$V(r, s) + V(s, t) = V(s, t) \qquad \text{if } s < r < t, \tag{5}$$

and, second, that $V(s, t)$ is separately continuous in s and t. This follows from (5), from the monotonicity of V, and from the continuity of $t \mapsto A(t) \in \mathscr{L}(Y, X)$. The reader mainly interested in the case where $t \mapsto A(t) \in \mathscr{L}(Y, X)$ is Lipschitz with some constant L may replace $V(t, s)$ if $t > s$ by $CL(t - s)$ in all the following.

Step 1: The constant C in (4) may be chosen in such a way that for all $s, t \in [0, T]$ and all $y \in D$,

$$\|y\|_t \leq e^{V(t,s)}\|y\|_s.$$

Proof. By the continuity of $t \mapsto \underline{A}(t) \in \mathscr{L}(Y, X)$, the map $t \mapsto \underline{A}(t)^{-1} \in \mathscr{L}(X, Y)$ is continuous and hence $C := \sup_{s \in I} \|\underline{A}(s)^{-1}\|_{X,Y} < \infty$. In view of $\|y\|_t \leq \|\underline{A}(t)\underline{A}(s)^{-1}\|\|y\|_s$, Step 1 follows from

$$\begin{aligned}\|\underline{A}(t)\underline{A}(s)^{-1}\| &= \|1 + (\underline{A}(t) - \underline{A}(s))\underline{A}(s)^{-1}\| \\ &\leq 1 + \|\underline{A}(t) - \underline{A}(s)\|_{Y,X}\|\underline{A}(s)^{-1}\|_{X,Y} \leq 1 + V(t, s) \leq e^{V(t,s)}.\end{aligned}$$

We now choose a sequence of partitions π_n of $[0,T]$ with the property that the mesh size of π_n vanishes in the limit $n \to \infty$. Given $t \in [0,T]$ and $n \in \mathbb{N}$ we use t_n to denote the largest element of π_n less than or equal to t. The smallest element of π_n larger than t_n is denoted t_n^+, the largest one smaller than t_n is denoted t_n^-. We thus have $t_n^- < t_n < t_n^+$ and

$$t_n \le t < t_n^+.$$

Note that the points t_n and $t_n^\pm$ are functions of both t and n. We define $U_n(t,s)$ for $t > s$ by

$$U_n(t,s) := e^{A(t_n)(t-t_n)} e^{A(t_n^-)(t_n - t_n^-)} \cdots e^{A(s_n)(s_n^+ - s)}$$

and $U_n(s,t) := U_n(t,s)^{-1}$. Note that $\|U_n(t,s)\| \le e^{\omega|t-s|}$ by assumption (3).

Step 2: For all $t > s$, $n \in \mathbb{N}$, and $y \in D$,

$$\begin{aligned} &\|U_n(t,s)y\|_t \le e^{V(t,s)+2V(s,s_n)+\omega(t-s)} \|y\|_s \\ \text{and} \quad &\|U_n(s,t)y\|_s \le e^{V(t,s)+2V(s,s_n)+\omega(t-s)} \|y\|_t. \end{aligned}$$

In particular, $\|U_n(t,s)\|_{Y,Y} < M$ for all $s,t \in [0,T]$ and all $n \in \mathbb{N}$.

Proof. With the help of Step 1 we pass from $\|\cdot\|_t$ to $\|\cdot\|_{t_n}$, then from $\|\cdot\|_{t_n}$ to $\|\cdot\|_{t_n^-}$ and so on, where in each step we use that $e^{A(t)\tau}$ is a quasi-contraction in Y_t satisfying (3) for any $t \in [0,T]$. In this way we arrive at

$$\|U_n(t,s)y\|_t \le e^{V(t,s_n)+\omega(t-s)} \|y\|_{s_n},$$

which, using Step 1 again, leads to the first of the asserted inequalities. The second one is proved analogously and the uniform bound on $\|U_n(t,s)\|_{Y,Y}$ now follows from Step 1 and the compactness of $[0,T]$.

Step 3: For all $x \in X$, the limit $U(t,s)x := \lim_{n\to\infty} U_n(t,s)x$ exists uniformly in $s,t \in [0,T]$. It defines an evolution system $U(t,s)$.

Proof. For any $y \in Y$ the map $\tau \mapsto U_m(t,\tau)U_n(\tau,s)y$ is piecewise continuously differentiable with possible jumps in the derivative at the partition points from $\pi_m \cup \pi_n$. It follows that

$$\begin{aligned} U_n(t,s)y - U_m(t,s)y &= U_m(t,\tau)U_n(\tau,s)y\big|_{\tau=s}^{\tau=t} \\ &= \int_s^t U_m(t,\tau)\big(A(\tau_n) - A(\tau_m)\big)U_n(\tau,s)y\, d\tau. \end{aligned}$$

By Step 2 we conclude

$$\|U_n(t,s)y - U_m(t,s)y\| \le \int_0^1 e^{\omega|t-\tau|} \|A(\tau_n) - A(\tau_m)\|_{Y,X} M \|y\|_Y \, d\tau \to 0 \qquad (n,m \to \infty)$$

by the continuity of $\tau \mapsto A(\tau) \in \mathscr{L}(Y,X)$. The assertion now follows from the density of $Y \subset X$ and from the uniform boundedness $\|U_n(t,s)\| \le e^{\omega|t-s|}$. It follows that $(t,s) \mapsto U(t,s)x$ is continuous and the property (ii) of evolution systems is inherited from $U_n(t,s)$ as well.

Step 4: $U(t,s)D \subset D$, and for all $y \in D$ and $s,t \in [0,T]$,

$$\|U(t,s)y\|_t \le e^{V(t,s)+\omega|t-s|} \|y\|_s.$$

Proof. Let $y \in D$. By Step 2 the sequence $(U_n(t,s)y)$ is bounded in Y_t and by Step 3, $U_n(t,s)y \to U(t,s)y$ in X. Since Y_t is reflexive it follows that $U(t,s)y \in D$ and that $U_n(t,s)y \to U(t,s)y$ weakly in Y_t. Therefore, by the estimates of Step 2,

$$\|U(t,s)y\|_t \leq \liminf_{n\to\infty} \|U_n(t,s)y\|_t \leq e^{V(t,s)+\omega|t-s|}\|y\|_s,$$

where $V(s,s_n) \to 0$ as $n \to \infty$ was used.

Step 5: For all $y \in D$ the map $t \mapsto U(t,s)y$ is differentiable in the norm of X and

$$\frac{d}{dt}U(t,s)y = A(t)U(t,s)y.$$

Proof. In view of $U(t,s)Y \subset Y$ and $U(t+h,s) = U(t+h,t)U(t,s)$, see Step 4, it suffices to prove the assertion for $s = t$. For any $y \in D$,

$$\begin{aligned} U(t+h,t)y - e^{A(t)h}y &= \lim_{n\to\infty} e^{A(t)(h+t-\tau)}U_n(\tau,t)y\Big|_{\tau=t}^{\tau=t+h} \\ &= \lim_{n\to\infty}\int_t^{t+h} e^{A(t)(h+t-\tau)}\big(A(\tau_n)-A(t)\big)U_n(\tau,s)y\,d\tau. \end{aligned}$$

By Step 2 it thus follows that

$$\begin{aligned} &\Big\|\frac{1}{h}(U(t+h,t)y - e^{A(t)h}y)\Big\| \\ &\leq \lim_{n\to\infty}\frac{1}{|h|}\left|\int_t^{t+h} e^{\omega|t+h-\tau|}\|A(\tau_n)-A(t)\|_{Y,X}\,d\tau\right| M\|y\|_Y \\ &= \frac{1}{|h|}\left|\int_t^{t+h} e^{\omega|t+h-\tau|}\|A(\tau)-A(t)\|_{Y,X}\,d\tau\right| M\|y\|_Y \to 0 \qquad (h\to 0) \end{aligned}$$

by the continuity of $\tau \mapsto A(\tau) \in \mathscr{L}(Y,X)$. Since $(e^{A(t)h}y - y)/h \to A(t)y$ as $h \to 0$, Step 5 now follows.

Step 6: For all $y \in D$ the map $t \mapsto A(t)U(t,s)y$ is continuous in the norm of X.

Proof. By the continuity of $t \mapsto A(t) \in \mathscr{L}(Y,X)$ it suffices to show that $t \mapsto U(t,s)y$ is continuous in the norm of Y. To this end it suffices to show that $\lim_{h\to 0} U(t+h,t)y = y$ in the norm of Y or, equivalently, in the norm of Y_t. Since $U(t+h,t)y \to y$ in X and since $h \mapsto U(t+h,t)y$ is bounded in Y_t, see Step 1 and Step 4, it follows that $U(t+h,t)y \to y$ weakly in Y_t. See the proof of Step 4 for a similar argument. Therefore,

$$\begin{aligned} \|y\|_t &\leq \liminf_{h\to 0}\|U(t+h,t)y\|_t \leq \limsup_{h\to 0}\|U(t+h,t)y\|_t \\ &\leq \limsup_{h\to 0} e^{V(t+h,t)}\|U(t+h,t)y\|_{t+h} \leq \limsup_{h\to 0} e^{2V(t+h,t)+\omega|h|}\|y\|_t = \|y\|_t. \end{aligned}$$

The weak convergence $U(t+h,t)y \to y$ in Y_t and the convergence of the norms implies norm convergence in Y_t by the uniform convexity. □

Remark 2.4. 1. At the end of the introduction we pointed out two results of Kato that are closely related to Theorem 2.1. There are two further prominent results in the literature on well-posedness, both due to Kato again, that can be compared

to Theorem 2.1: By a simple corollary of Theorem 1 of [8], see Theorem 2.1.9 in [15], it suffices to assume that

$$t \mapsto A(t) \in \mathscr{L}(Y, X) \tag{6}$$

satisfies a certain $W_*^{1,1}$-regularity condition. This condition implies that (6) is absolutely continuous and hence continuous and of bounded variation. The condition $\partial_t A(t) \in L_*^\infty([0,T], \mathscr{L}(Y,X))$ used by Kato in [5], implies that (6) is Lipschitz and hence continuous and of bounded variation.

2. If $A(t)$ was assumed to be the generator of a semigroup, rather than a group, in Theorem 2.1, then the arguments of our proof above still establish existence of a (unique) evolution system $U(t,s)$ defined on the triangle $0 \leq s \leq t \leq T$ such that

$$\partial_t^+ U(t,s)y = A(t)U(t,s)y,$$

where $t \mapsto A(t)U(t,s)y$ is right-continuous and ∂_t^+ denotes the derivative from the right. Moreover, $\partial_t U(t,s)y = A(t)U(t,s)y$ except possibly for a countable set of t-values depending on y and s (see the proof of Theorem 5.2 of [7]).

3. In the case where the first or higher, suitably defined commutators of the operators $A(t)$ at distinct times are scalars, the continuity of the map (2), along with strong continuity of the commutators, is sufficient for well-posedness [4, 9, 14].

4. In the case where X is a Hilbert space there are formal similarities between our Theorem 2.1 and the Theorem C.2 of Ammari and Breteaux [1]. In [1] the case of skew-selfadjoint generators with time-independent *form domains* is considered and a notion of well-posedness in a weak sense is established.

In the remainder of this paper we specialize to operator families of the form $A(t) = A_0 + B(t)$ where A_0 is the generator of a C_0-group in X, $B(t) \in \mathscr{L}(X)$ and $t \mapsto B(t)$ is strongly continuous. Suppose that the evolution system $U(t,s)$ for A exists. Then, for all $y \in D(A_0)$,

$$U(t,s)y = e^{A_0(t-s)}y + \int_s^t d\tau e^{A_0(t-\tau)} B(\tau)U(\tau,s)y$$

which, by assumption on $B(t)$, may be iterated indefinitely into a convergent Dyson series [13]. For the evolution system in the interaction picture we obtain

$$\begin{aligned} e^{-A_0 t} U(t,s) e^{A_0 s} y = y &+ \int_s^t d\tau_1 \tilde{B}(\tau_1) y \\ &+ \int_s^t d\tau_1 \int_s^{\tau_1} d\tau_2 \tilde{B}(\tau_1)\tilde{B}(\tau_2) y + \ldots, \end{aligned} \tag{7}$$

where $\tilde{B}(\tau)y := e^{-A_0\tau} B(\tau) e^{A_0\tau}$. Consequently, the operator family $U(t,s)$ defined by (7) is the only candidate for the evolution system generated by $A(t) = A_0 + B(t)$.

The following theorem is now an immediate corollary of the previous one and Theorem 3.1.1 from [10]. In the uniformly convex case it improves on a similar result due to Phillips: in Theorem 6.2 of [11] it is assumed that $t \mapsto B(t)$ is strongly continuously differentiable.

Theorem 2.5. *Suppose X is a uniformly convex Banach space and that $A(t) = A_0 + B(t)$ for $t \in [0, T]$, where $A_0 : D \subset X \to X$ is the generator of a strongly continuous quasi-contractive group in X and $B(t) \in \mathscr{L}(X)$. If $t \mapsto B(t)y$ is continuous for all $y \in X$ and Lipschitz for all $y \in D$, then there exists a unique evolution system $U(t, s)$ for A on D and $e^{-A_0 t}U(t, s)e^{A_0 s}$ is given by the Dyson series* (7).

The following examples show that the assumptions of uniform convexity and Lipschitz continuity in this theorem and hence in the Theorem 2.1 cannot be weakened in an essential way.

Example 1. Let $X = C_0(\mathbb{R})$ be the Banach space of bounded and continuous functions vanishing at infinity, the norm being the usual maximum norm. This space is not uniformly convex. Let $e^{A_0 t}$ be the strongly continuous group in X defined by left translations, that is, $e^{A_0 t}x(\xi) = x(\xi + t)$. We define $A(t) : D \subset X \to X$ for $t \in [0, 1]$ by $D = D(A_0)$ and

$$A(t) = A_0 + B(t), \qquad B(t) = e^{A_0 t}Be^{-A_0 t}$$

where B denotes multiplication with the following bounded function $f : \mathbb{R} \to [0, 1]$: we choose $f(\xi) = 0$ for $\xi \leq 0$, $f(\xi) = \xi$ for $\xi \in [0, 1]$ and $f(\xi) = 1$ for $\xi \geq 1$. Then $B(t)$ is multiplication with the function $\xi \mapsto f(\xi + t)$ and from the fact that f is Lipschitz it is easy to check that $t \mapsto B(t)$ is strongly Lipschitz. If an evolution system $U(t, s)$ for A on D existed, then it would be given by the Dyson series (7) and since $\tilde{B}(t) = B$ it would follow that

$$U(t, 0) = e^{A_0 t}e^{Bt}. \tag{8}$$

Since $D = D(A_0)$ is left invariant by $e^{-A_0 t}$ it would follow that $e^{Bt}D(A_0) \subset D(A_0)$. But $D(A_0) = \{y \in C^1(\mathbb{R}) \mid y, y' \in X\}$ and the operator e^{Bt} acts as multiplication with the non-differentiable function $e^{f(\xi)t}$. Hence $e^{Bt}D(A_0) \not\subset D(A_0)$ and we have a contradiction. Therefore an evolutions system U for A on D cannot exist.

Example 2. For this example we adopt all elements of Example 1 with two exceptions: now $X = L^2(\mathbb{R})$ and f denotes multiplication with the bounded function $f = ig$, where $g : \mathbb{R} \to \mathbb{R}$ is the Weierstraß function

$$g(\xi) = \sum_{n=1}^{\infty} 2^{-n} \cos(2^n \xi).$$

This function is Hölder continuous of degree α for all $\alpha < 1$ and nowhere differentiable. See [18], Theorem II.4.9, including the proof, and the remark after Theorem II.4.10. It easily follows that $t \mapsto B(t)$ is strongly Hölder continuous of degree α for all $\alpha < 1$. As in Example 1 we argue that $e^{Bt}D(A_0) \subset D(A_0)$ if the evolution system U for A existed. But e^{Bt} acts by multiplication with $\xi \mapsto e^{f(\xi)t}$, which is nowhere differentiable, and $D(A_0) = H^1(\mathbb{R})$ whose elements are differentiable almost everywhere. We have a contradiction and hence an evolution system U for A on D cannot exist. Note that $A(t)$ is skew-selfadjoint in this example.

Acknowledgement: J. S. would like to thank Roland Schnaubelt for interesting discussions and for pointing out [12] to us. His work was supported by the *Deutsche Forschungsgemeinschaft (DFG)* through the Research Training Group 1838.

References

[1] Z. Ammari and S. Breteaux. Propagation of chaos for many-boson systems in one dimension with a point pair-interaction. *Asymptot. Anal.*, 76(3-4):123–170, 2012.

[2] J. R. Dorroh. A simplified proof of a theorem of Kato on linear evolution equations. *J. Math. Soc. Japan*, 27(3):474–478, 1975.

[3] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[4] Jerome A. Goldstein. *Semigroups of linear operators and applications*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1985.

[5] T. Kato. *Abstract differential equations and nonlinear mixed problems*. Lezioni Fermiane. [Fermi Lectures]. Scuola Normale Superiore, Pisa, 1985.

[6] Tosio Kato. Integration of the equation of evolution in a Banach space. *J. Math. Soc. Japan*, 5:208–234, 1953.

[7] Tosio Kato. Linear evolution equations of “hyperbolic” type. *J. Fac. Sci. Univ. Tokyo Sect. I*, 17:241–258, 1970.

[8] Tosio Kato. Linear evolution equations of “hyperbolic” type. II. *J. Math. Soc. Japan*, 25:648–666, 1973.

[9] Gregor Nickel and Roland Schnaubelt. An extension of Kato’s stability condition for nonautonomous Cauchy problems. *Taiwanese J. Math.*, 2(4):483–496, 1998.

[10] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[11] R. S. Phillips. Perturbation theory for semi-groups of linear operators. *Trans. Amer. Math. Soc.*, 74:199–221, 1953.

[12] Jan Prüss and Roland Schnaubelt. Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time. *J. Math. Anal. Appl.*, 256(2):405–430, 2001.

[13] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[14] J. Schmid. Well-posedness of non-autonomous linear evolution equations for generators whose commutators are scalar. *J. Evol. Equ.*, DOI 10.1007/s00028-015-0291-5. arXiv:1411.0857, 2014.

[15] Jochen Schmid. *Adiabatic theorems for general linear operators and well-posedness of linear evolution equations.* PhD thesis, University of Stuttgart, 2015.

[16] Hiroki Tanabe. *Equations of evolution*, volume 6 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass., 1979. Translated from the Japanese by N. Mugibayashi and H. Haneda.

[17] Kenji Yajima. Schrödinger equations with time-dependent unbounded singular potentials. *Rev. Math. Phys.*, 23(8):823–838, 2011.

[18] A. Zygmund. *Trigonometric series. Vol. I, II.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1979 edition.